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A Note on Vanishing of the Functor Ext^1 for Köthe Spaces

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In this note we study the relationship between the vanishing of $\mathrm{Ext}^1(\lambda(A), \lambda(A))$ and the existence of a regular basis in the Köthe space $\lambda(A)$. We construct an example of a nuclear Köthe space $\lambda(A)$ with no regular basis and such that $\mathrm{Ext}^1(\lambda(A), \lambda(A)) = 0$. Then we show that for some classes of Köthe spaces $\lambda(A)$, the vanishing of $\mathrm{Ext}^1(\lambda(A), \lambda(A))$ yields a regular basis for $\lambda(A)$.

Introduction

In [7], the Köthe spaces $\lambda(A)$ with property (DN) and satisfying the condition $\mathrm{Ext}^1(\lambda(A), \lambda(A)) = 0$ have been completely characterized. As a by-product of this characterization it follows that such spaces always have regular bases. It was believed that in this result the condition (DN) was superfluous. In this note we give an example of a nuclear Köthe space $\lambda(A)$ such that $\mathrm{Ext}^1(\lambda(A), \lambda(A)) = 0$, but $\lambda(A)$ has no regular basis. However we show that if $\lambda(A)$ is a direct sum or tensor product of two Dragilev spaces (defined by functions with comparable growth rates) and $\mathrm{Ext}^1(\lambda(A), \lambda(A)) = 0$, then $\lambda(A)$ has a regular basis. We also show that the same conclusion holds if $\lambda(A)$ is a direct sum or tensor product of a (DN) space and a regular space with property $(\overline{\Omega})$.

Notation and terminology

For the terminology not defined here we refer the reader to Dubinsky [4] and Vogt [11].

In the sequel we shall assume that all Köthe spaces $\lambda(A)$ are Fréchet and Schwartz, i.e. $A = (a_i^k)$ is such that for all $i, k \in \mathbb{N} = \{1, 2, \ldots\}$ we have $0 < a_i^k \leq a_i^{k+1}$ and for all k, $\lim_i a_i^k / a_i^{k+1} = 0$.

The condition (S_2^*) (called (S^*) in [8]) was defined in [11] by Vogt. Let $\lambda(A)$ and $\lambda(B)$ be two Köthe spaces. The pair $(\lambda(A), \lambda(B))$ is said to satisfy the condition (S_2^*) (briefly $(\lambda(A), \lambda(B)) \in (\mathrm{S}_2^*)$) if the following holds:

$$\forall \mu\ \exists n_0, k\ \forall K, m\ \exists n, S\ \forall i, j: \quad \frac{a_i^m}{b_j^k} \leq S \max\left(\frac{a_i^n}{b_j^K}, \frac{a_i^{n_0}}{b_j^{\mu}}\right).$$

It is easy to see that this condition is equivalent to the following

$$\forall \mu\, \exists k\, \forall K\, \exists n\, \exists i_0, j_0\, \forall i \geq i_0,\ \forall j \geq j_0 : \quad \frac{a_i^K}{b_j^k} \leq \max\left(\frac{a_i^n}{b_j^K}, \frac{a_i^k}{b_j^\mu}\right).$$

It was shown in [11] and [8] that $(\lambda(A), \lambda(B))$ satisfies (S_2^*) if and only if $\mathrm{Ext}^1(\lambda(A), \lambda(B)) = 0$ if and only if every short exact sequence $0 \to \lambda(B) \to F \to \lambda(A) \to 0$ of Fréchet spaces splits.

The conditions (DN) and $(\overline{\Omega})$ were defined by Vogt [10] and Wagner [12] to characterize subspaces and quotients of nuclear and stable power series spaces of infinite type and finite type respectively. Here we state Köthe space versions of these conditions which in this case turn out to be the same as the conditions (d_1) and (d_2) of Dragilev [3]. A Köthe space $\lambda(A)$ is said to have the property (DN) if $\exists n_0\, \forall m\, \exists n, C > 0\, \forall i : \ (a_i^m)^2 \leq C a_i^{n_0} a_i^n$,
$(\overline{\Omega})$ if $\forall p\, \exists q\, \forall k\, \exists C > 0\, \forall i : \ C(a_i^q)^2 \geq a_i^k a_i^p$.

A Köthe space $\lambda(A)$ is said to be *regular* if for all i and k, $a_i^{k+1}/a_i^k \leq a_{i+1}^{k+1}/a_{i+1}^k$ and it is called *pseudo regular* if

$$\forall p\, \exists q\, \forall r > q\, \exists s > p\, \exists M > 0 : i \leq j \Rightarrow \frac{a_i^r}{a_i^q} \leq M \frac{a_j^s}{a_j^p}.$$

Regularity was defined in [3] and pseudo regularity in [2]. It is obvious that pseudo regularity is weaker than regularity (it is not known whether it is strictly weaker.)

In [2], it was shown that if $\lambda = \lambda(A)$ is a nuclear, pseudo regular Köthe space, then $\delta(\lambda) = \lambda \cdot \lambda^\times$. Here $\delta(\lambda)$ denotes the diametral dimension of λ (see [1]) and $\lambda^\times$ is the Köthe dual (or α-dual) of λ. It is well-known that if λ is nuclear, then $\delta(\lambda) \subset \delta(s)$ where s denotes the space of rapidly decreasing sequences, i.e. $s = \lambda(B)$ where $b_i^k = \exp k \log i$ and $\delta(s) = s$. Hence for a nuclear, pseudo regular Köthe space $\lambda = \lambda(A)$ we have $\lambda \cdot \lambda^\times \subset s$ and this last condition is equivalent to the following:

$$\forall k\, \forall \ell\, \exists m : \ \sup \frac{a_i^k e^{\ell \log i}}{a_i^m} < \infty.$$

(See proof of (3.3) and (5.1) (i) in [9].)

$L_f(a,r)$-spaces (also called *Dragilev spaces*) were introduced in [3]. Let $f : \mathbf{R} \to \mathbf{R}$ be an odd, strictly increasing, logarithmically convex function (i.e. $\log \circ f \circ \exp$ is convex), $a = (a_i)$ be an exponent sequence, i.e. $0 < a_i \nearrow \infty$, $r_k \nearrow r$ where $r = -1, 0, 1$ or ∞. Then $L_f(a,r)$ is defined as the Köthe space $\lambda(A)$ generated by the matrix $a_i^k = \exp f(r_k a_i)$. If f is logarithmically convex, then it satisfies one of the following conditions:
$\forall a > 1,\ \lim_{x\to\infty} f(ax)/f(x) = \infty$. In this case f is called *rapidly increasing,*
$\forall a > 1,\ \lim_{x\to\infty} f(ax)/f(x) < \infty$. In this case f is called *slowly increasing.*
When f is slowly increasing $L_f(a,r)$ is isomorphic to a power series space and without loss of any generality we can take $f =$ identity.

If f and g are two functions as in the definition of Dragilev spaces, we write $f \succ g$ or $g \prec f$ if $g^{-1}f$ is a rapidly increasing function,
$f \approx g$ if $g^{-1}f$ and $f^{-1}g$ are both logarithmically convex and slowly increasing.

The example

First we construct a nuclear Köthe space $\lambda(A)$ with no pseudo regular basis and with the property that $\mathrm{Ext}^1(\lambda(A), \lambda(A)) = 0$.

Proposition 1. *Let (α_i) be any sequence such that $0 < \alpha_i \nearrow \infty$ and the set of finite limit points of the set $\{\alpha_i/\alpha_j : i, j \in \mathbb{N}\}$ is bounded. Let $\mathbb{N} = \cup_{u=1}^{\infty} \mathbb{N}_u$ be a partition of $\mathbb{N}$ into pairwise disjoint infinite sets $\mathbb{N}_u$. Let $r_k \nearrow \infty$* and *$\sigma_k \searrow 0$. For each $i \in \mathbb{N}$, let $u = u(i)$ be the unique index such that $i \in \mathbb{N}_u$. Let*

$$\rho_i^k = \begin{cases} r_k & \text{if } u \leq k \\ -\sigma_k & \text{if } k < u \end{cases}$$

and set $a_i^k = \exp \rho_i^k \alpha_i$. Then $\mathrm{Ext}^1(\lambda(A), \lambda(A)) = 0$.

Proof. $\mathrm{Ext}^1(\lambda(A), \lambda(A)) = 0$ if and only if $\forall \mu\ \exists k\ \forall K\ \exists n\ \exists i_0, j_0 : \forall i \geq i_0$, $\forall j \geq j_0$

$$\text{either (I):} \quad \frac{\alpha_j}{\alpha_i} \leq \frac{\rho_i^n - \rho_i^K}{\rho_j^n - \rho_j^k} \quad \text{or (II):} \quad \frac{\rho_i^K - \rho_i^k}{\rho_j^k - \rho_j^\mu} \leq \frac{\alpha_j}{\alpha_i}$$

holds.

Let $0 < a \leq A < \infty$ be such that if α is any finite non-zero limit point of the set $\{\alpha_i/\alpha_j : i, j \in \mathbb{N}\}$ then $a \leq \alpha \leq A$.

Given μ, we choose k so that

$$\max\left(\frac{\sigma_k}{r_k - r_\mu}, \frac{\sigma_k}{r_k + \sigma_\mu}, \frac{\sigma_k}{\sigma_\mu - \sigma_k}\right) < a.$$

Given K, we choose n so that

$$A < \min\left(\frac{r_n - r_K}{r_K - r_k}, \frac{r_n - r_K}{r_K + \sigma_k}, \frac{r_n - r_K}{\sigma_k - \sigma_K}, \frac{r_n + \sigma_K}{r_K - r_k}, \frac{r_n + \sigma_K}{r_K + \sigma_k}, \frac{r_n + \sigma_K}{\sigma_k - \sigma_K}\right).$$

Given i and j, assume $i \in \mathbb{N}_u$ and $j \in \mathbb{N}_v$. We have several possibilities for the positions of u and v. If $u \leq k$, $v \leq \mu$, then

$$\text{(I):} \quad \frac{\alpha_j}{\alpha_i} \leq \frac{r_n - r_K}{r_K - r_k}, \quad \text{(II):} \quad \frac{r_K - r_k}{r_k - r_\mu} \leq \frac{\alpha_j}{\alpha_i}$$

In general if $u \leq n$ and v is arbitrary we have

$$\text{(I):} \quad \frac{\alpha_j}{\alpha_i} \leq \frac{r_n - \cdots}{\cdots}$$

and by our choice of n for all values of i and j for which α_j/α_i accumulate about a finite limit point, (I) holds. For the others (II) holds.

If $n < u$, then (II) becomes $(\sigma_k - \sigma_K)/(\cdots) \leq \alpha_j/\alpha_i$ and by our choice of k, for all values of i and j for which α_j/α_i accumulate about a limit point which is bigger than or equal to a, (II) holds. For the other values of i and j, (I) holds. $\square$

Example. Let (i_n) be a subsequence of $\mathbb{N}$ such that $\lim_{n\to\infty} \log i_{n+1}/\log i_n = \infty$. If (i_n) is such a sequence then for all large n we have $i_n \geq n^n$. Let $i_0 = 0$ and define

$$\alpha_i = \alpha_{i_n} = \log i_n \quad \text{for} \quad i_{n-1} + 1 \leq i \leq i_n, \quad n = 1, 2, \ldots.$$

Let

$$\mathbb{N}_1 = \{i : i \neq i_n,\ n = 1, 2, \ldots\}, \quad \mathbb{N}_1' = \{i_n : n = 1, 2, \ldots\}$$

and partition $\mathbb{N}_1'$ into pairwise disjoint infinite sets $\mathbb{N}_2 \cup \mathbb{N}_3 \cup \ldots$

Since the set $\{\alpha_i/\alpha_j\}$ has only three limit points (namely 0, 1 and ∞), the Köthe space $\lambda(A)$ defined as in Proposition 1 has the property that $\mathrm{Ext}^1(\lambda(A), \lambda(A)) = 0$. Now we show that $\lambda(A)$ is nuclear. For this observe the following:

(i) $\forall c > 0$, $\sum_{n=1}^{\infty} \frac{1}{e^{c\alpha_{i_n}}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{cn}} < \infty$.

(ii) $\forall c > 1$, $\sum_{i=1}^{\infty} \frac{1}{e^{c\alpha_i}} = \sum_{n=1}^{\infty} \sum_{i=i_{n-1}+1}^{i_n} \frac{1}{e^{c\alpha_i}} = \sum_{n=1}^{\infty} \frac{i_n - i_{n-1}}{e^{c\alpha_{i_n}}} \leq \sum_{n=1}^{\infty} \frac{i_n}{i_n^c} < \infty$.

Given k we find ℓ such that $r_\ell > r_k + 1$. If $i \in \mathbb{N}_u$ we have

$$\frac{a_i^k}{a_i^\ell} = \begin{cases} \frac{1}{e^{(r_\ell - r_k)\alpha_i}} & \text{if } u \leq k \\ \frac{1}{e^{(r_\ell + \sigma_k)\alpha_i}} & \text{if } k < u \leq \ell \\ \frac{1}{e^{(\sigma_k - \sigma_\ell)\alpha_i}} & \text{if } \ell < u. \end{cases}$$

Then for $u \geq 2$ we have

$$\frac{a_i^k}{a_i^\ell} \leq \frac{1}{e^{(\sigma_k - \sigma_\ell)\alpha_i}}$$

and so

$$\sum_{i=1}^{\infty} \frac{a_i^k}{a_i^\ell} \leq \sum_{i \in \mathbb{N}_1} \frac{1}{e^{(r_\ell - r_k)\alpha_i}} + \sum_{n=1}^{\infty} \frac{1}{e^{(\sigma_k - \sigma_\ell)\alpha_{i_n}}} < \infty.$$

Finally we show that $\lambda(A)$ does not have a pseudo regular basis. If it did we would have

$$\forall k\ \forall \ell\ \exists m : \ \sup \frac{a_i^k e^{\ell \log i}}{a_i^m} = C < \infty.$$

Let $k = \ell = 1$. Given any $m > k$, let $u > m$. Then

$$\sup_{i \in \mathbb{N}_u} \frac{a_i^k e^{\ell \log i}}{a_i^m} = \sup_{i \in \mathbb{N}_u} e^{\ell \log i - (\sigma_k - \sigma_m)\alpha_i} = \sup_{i \in \mathbb{N}_u} e^{(\ell - (\sigma_k - \sigma_m)) \log i} = \infty.$$

Some observations

Next we indicate some cases in which the vanishing of $\mathrm{Ext}^1(\lambda(A), \lambda(A))$ yields a regular basis for $\lambda(A)$.

Theorem 1. *Let $\lambda(A)$ and $\lambda(B)$ be Schwartz regular Köthe spaces with $\lambda(A)$ having property* (DN) *and $\lambda(B)$ having property* $(\overline{\Omega})$. *If* $\mathrm{Ext}^1(\lambda(B), \lambda(A)) = 0$, *then $\lambda(A) \oplus \lambda(B)$ and $\lambda(A)\hat{\otimes}_\pi \lambda(B)$ have regular bases.*

Proof. After a suitable standardization (see [8]) we may assume that

$$\frac{b_j^k}{a_i^k} < \max\left(\frac{b_j^{k-1}}{a_i^{k-1}}, \frac{b_j^{k+1}}{a_i^{k+1}}\right) \tag{1}$$

$$\left(a_i^k\right)^2 \leq a_i^{k-1} a_i^{k+1}, \quad a_i^1 = 1 \tag{2}$$

$$\left(b_j^k\right)^2 \geq b_j^{k-1} b_j^{k+1}, \quad b_j^1 = 1 \tag{3}$$

for all k, i and j.

Let (e_i) (resp. (f_j)) be the natural basis vectors in $\lambda(A)$ (resp. $\lambda(B)$). The rearrangement of the sequence of vectors $e_1, f_1, e_2, f_2, \ldots$ corresponding to a nondecreasing rearrangement of the sequence $a_1^2, b_1^2, a_2^2, b_2^2, \ldots$ is easily seen to produce a regular basis of $\lambda(A) \oplus \lambda(B)$.

Next we consider $\lambda(A)\hat{\otimes}_\pi \lambda(B)$ which is isomorphic to $\lambda(C)$ where $c_{i,j}^k = a_i^k b_j^k$. We let

$$I = \{(i,j) : a_i^2 \leq b_j^2\}, \quad J = \mathbf{N} \times \mathbf{N} \setminus I.$$

For $(i,j) \in I$ by (1) we have

$$\frac{a_i^{k+1}}{a_i^k} \leq \frac{b_j^{k+1}}{b_j^k} \quad \text{for all } k$$

hence because of (2) and (3)

$$\frac{a_i^{\ell+1}}{a_i^\ell} \leq \frac{b_j^{k+1}}{b_j^k} \quad \text{for all } \ell, k \tag{4}$$

and so

$$a_i^k \leq \frac{b_j^{2k}}{b_j^k} \quad \text{for all } k$$

and therefore

$$b_j^k \leq a_i^k b_j^k \leq b_j^{2k}.$$

For $(i,j) \in J$, because of (2) and (3) we have the reverse inequality for (4), hence

$$a_i^k \leq a_i^k b_j^k \leq a_i^{2k-1} b_i^1 = a_i^{2k-1}.$$

So $\lambda(C) \simeq \lambda(C_1) \oplus \lambda(C_2)$ where $\lambda(C_1)$ has property (DN), $\lambda(C_2)$ has property $(\overline{\Omega})$ and $\mathrm{Ext}^1(\lambda(C_2), \lambda(C_1)) = 0$. This leads us to the first case. □

Since for a Köthe space $\lambda(A)$ with (DN) from $\mathrm{Ext}^1(\lambda(A), \lambda(A)) = 0$ it follows that $\lambda(A)$ has a regular basis (see [7]), we immediately obtain the following Corollary.

Corollary. *Let $\lambda(A)$ and $\lambda(B)$ be Schwartz Köthe spaces with properties* (DN) *and* $(\overline{\Omega})$ *respectively. Assume* $\mathrm{Ext}^1(\lambda(A) \oplus \lambda(B), \lambda(A) \oplus \lambda(B)) = 0$ *and $\lambda(B)$ has a regular basis. Then $\lambda(A) \oplus \lambda(B)$ and $\lambda(A)\hat{\otimes}_\pi\lambda(B)$ have regular bases.*

If $E = L_f(a, r)$ and $F = L_g(b, s)$ are two Dragilev spaces such that $f \prec g$ or $f \succ g$ or $f \approx g$, then complete characterizations for $\mathrm{Ext}^1(E, F) = 0$ have been given by Hebbecker [5] (see also [6]). By using these characterizations and a proof similar to that of Theorem 1 we obtain the following.

Theorem 2. *Let $E = L_f(a, r)$ and $F = L_g(b, s)$ be two Dragilev spaces such that either $f \approx g$ or $f \succ g$ or $f \prec g$ and* $\mathrm{Ext}^1(E \oplus F, E \oplus F) = 0$. *Then $E \oplus F$ and $E\hat{\otimes}_\pi F$ have regular bases.*

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